Mass-Sheet Degeneracy in Gravitational Wave Lensing

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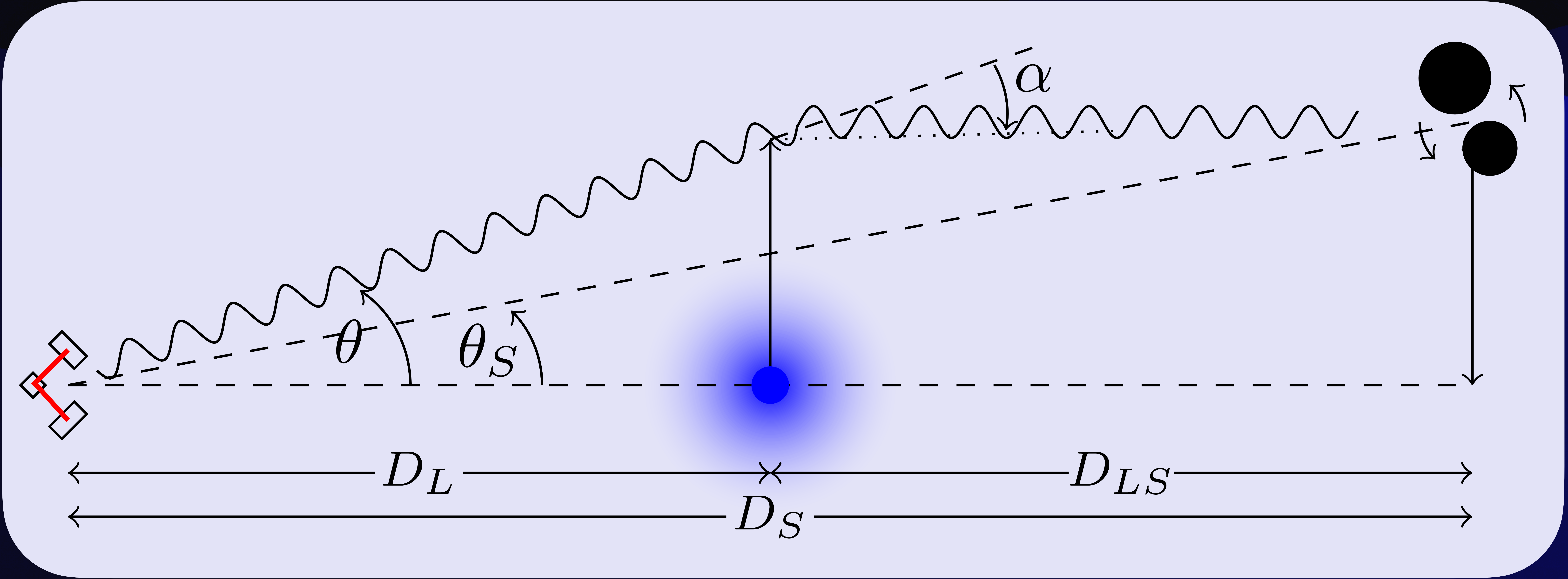
Strong group meeting

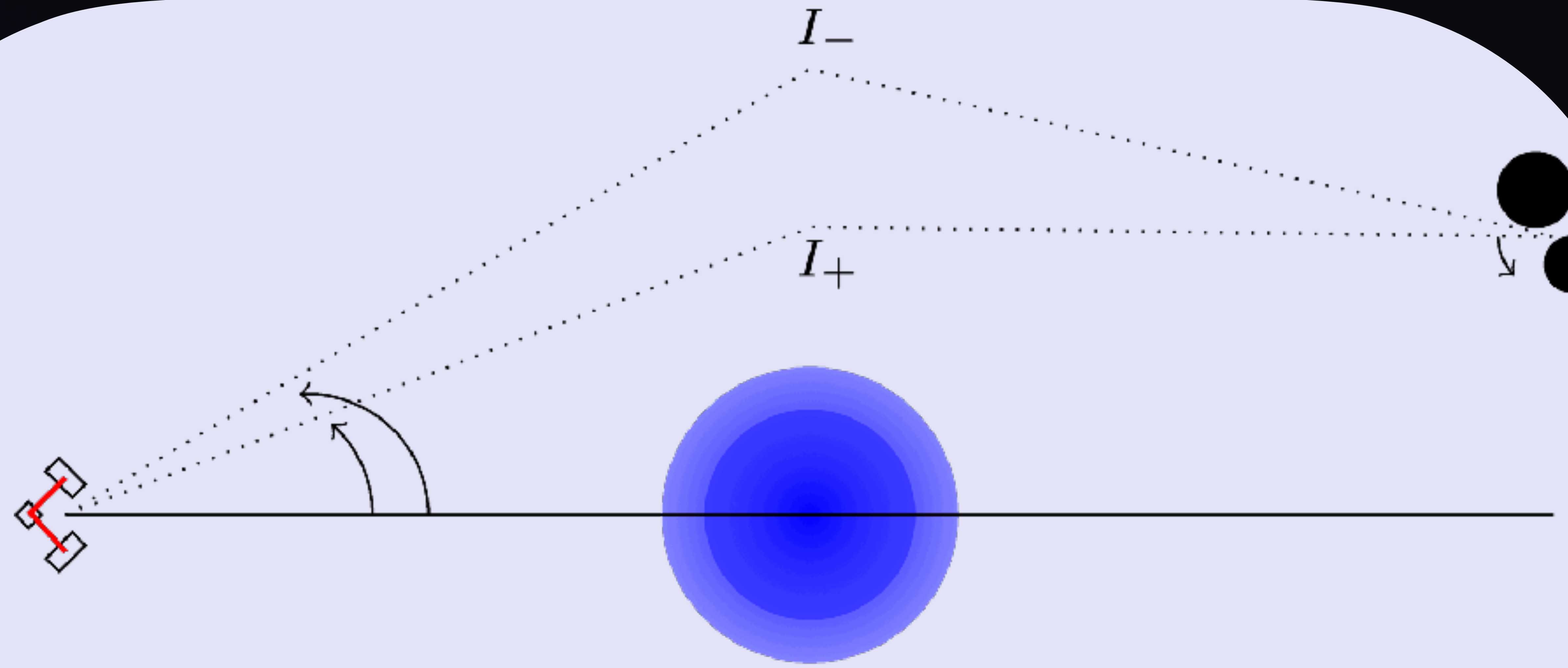
Gravitational Lensing of Gravitational Waves

Gravitational Lensing

EM: $10^3 < f < 10^{18}\ Hz$ | GW: $10^{-9} < f < 10^4\ Hz$



Geometrical-Optics vs Wave-Optics



GO approximation breaks when

$$M_{3D,L} \leq 10^5 M_\odot \left[\frac{(1+z_L)f}{\mathrm{Hz}}\right]^{-1}$$

$$f \cdot \Delta t \leq 1$$

Mass-Sheet Degeneracy

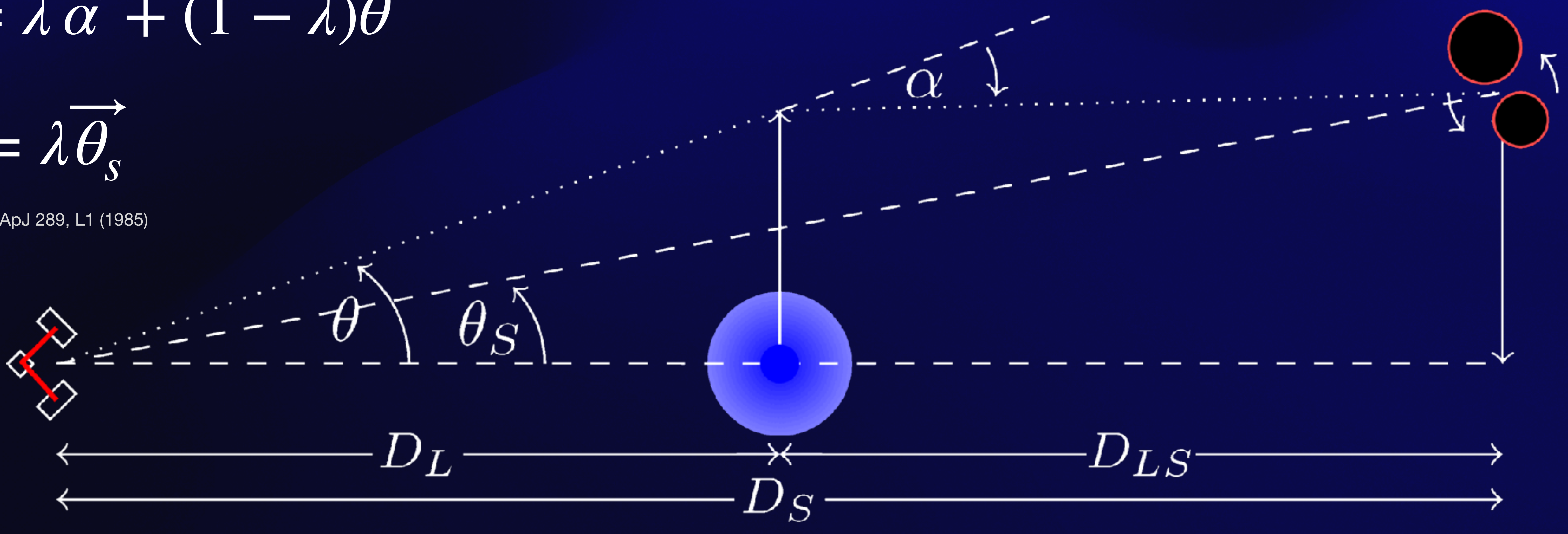
Mass-Sheet Degeneracy

- Scalings of lens mass:
  - $\kappa \rightarrow \kappa_\lambda = \lambda\kappa + (1 - \lambda)$
- Scaling angles:
  - $\vec{\alpha} \rightarrow \vec{\alpha}_\lambda = \lambda\vec{\alpha} + (1 - \lambda)\vec{\theta}$
  - $\vec{\theta}_s \rightarrow \vec{\theta}_{s,\lambda} = \lambda\vec{\theta}_s$

$\kappa = \Sigma/\Sigma_{cr}$

Σ - surface mass density

E. E. Falco, M. V. Gorenstein, and I. I. Shapiro, ApJ 289, L1 (1985)



Mass-Sheet Degeneracy

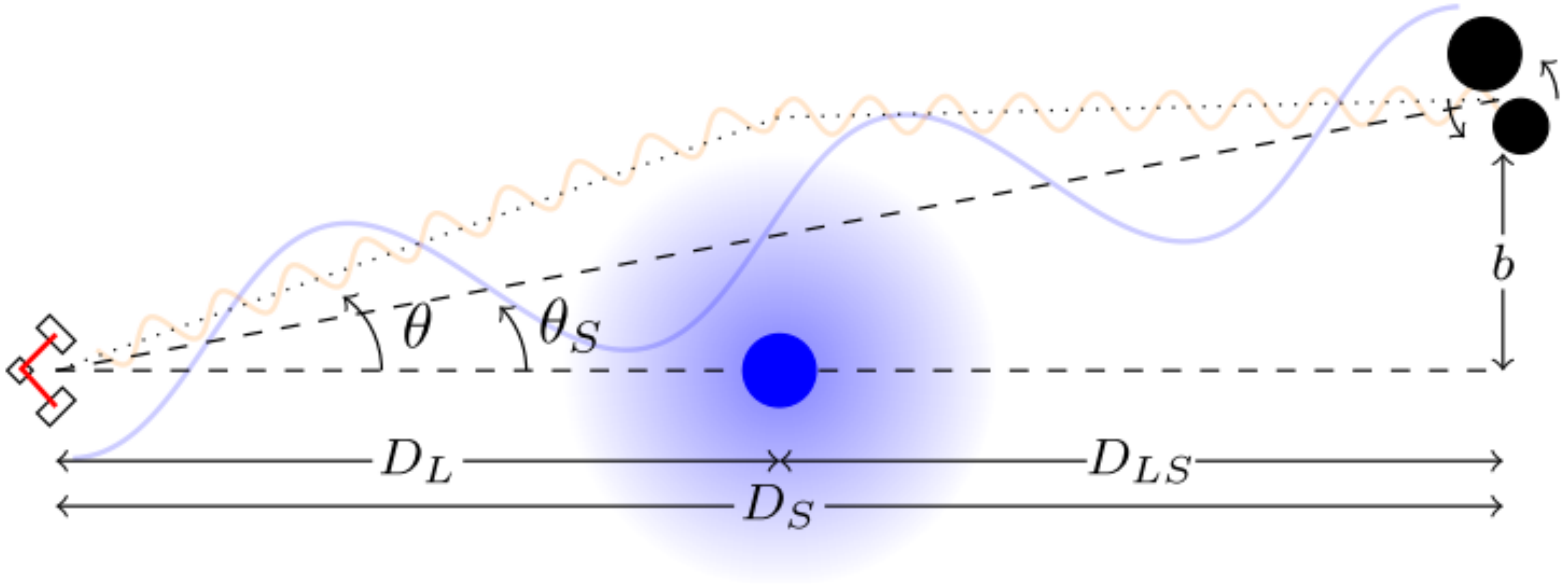
Why a problem?

- Observables are preserved!
- Problems: e.g. biased estimations of mass lens
- Biased estimation of cosmological parameter, e.g. H_0

Can we solve it?

- EM geometrical optics regime: multiple images; independent mass estimation of the lens (e.g. dynamics)
- EM wave optics regime: multiple lenses
- **In GW lensing: 1 image and 1 lens can break MSD!**

GW lensing



Time delay: $$t(\vec{\theta}, \vec{\theta}_S) = \frac{1+z_L}{c} \frac{D_L D_S}{D_{LS}} \left[\frac{1}{2}(\vec{\theta} - \vec{\theta}_S)^2 - \hat{\Psi}(\vec{\theta}) \right]$$

$$t_d = \frac{1+z_L}{c} \frac{D_L D_S \theta_E^2}{D_{LS}} \left(\frac{1}{2}|x-y|^2 - \Psi(x) \right)$$

$$(x = \theta/\theta_E, \; y = \theta_s/\theta_E)$$

θ_E : Einstein radius

$$t_d \propto \frac{1}{H_0}$$

Wave optics limit

The GW signal is amplified by the lens.

Amplification factor: $$F(w,y) = \frac{w}{2\pi i}\int d^2x \;\exp[iwT(x,y)]$$

w and T are dimensionless frequency and time delay.

$$F(w,y) = -iwe^{iwy^2/2}\int dx\; xJ_0(wxy)\exp\left\{iw\left[\frac{1}{2}x^2 - \Psi(x)\right]\right\}$$

For the point mass lens model, $\Psi(x) = \ln x$

$$F(\omega,y) = \left(-\frac{i\omega}{2}\right)^{1+\frac{i\omega}{2}}\exp\left(\frac{1}{2}i\omega y^2\right)\Gamma\left(-\frac{i\omega}{2}\right)1F_1\left(1-\frac{i\omega}{2};1;-\frac{i\omega}{2}y^2\right)$$

Mass-sheet degeneracy

The lens mass is added or subtracted by a constant mass sheet so that the convergence is transformed by

$$\kappa \rightarrow \kappa_\lambda = \lambda\kappa + (1-\lambda)$$

The deflection angle and the source position are also scaled by

$$\vec{\alpha} \rightarrow \vec{\alpha}_\lambda = \lambda\vec{\alpha} + (1-\lambda)\vec{\theta},$$

$$\vec{\theta}_S \rightarrow \vec{\theta}_{S,\lambda} = \lambda\vec{\theta}_S .$$

It leads to a transformation to the time delay.

$$t_d \rightarrow t_\lambda = \lambda t - \frac{\lambda(1-\lambda)}{2}\left(\frac{1+z_L}{c}\frac{D_L D_S \theta_E^2}{D_{LS}}\right)y^2$$

Mass-sheet degeneracy

The source position and the lens potential are transformed as

$$\vec{y} \rightarrow \vec{y}_\lambda = \lambda \vec{y};$$

$$\Psi(\vec{x}) \rightarrow \Psi_\lambda(\vec{x}) = \lambda \Psi(\vec{x}) + (1-\lambda)\frac{|\vec{x}|^2}{2};$$

The amplification factor is then given by

$$F_\lambda(w,y) = -iwe^{iw\lambda^2 y^2/2}\int_0^\infty dx\; xJ_0(\lambda wxy)\exp\left\{iw\lambda\left[\frac{x^2}{2} - \Psi(x)\right]\right\}$$

For the point mass lens model,

$$F_\lambda = \frac{1}{\lambda}\left(-\frac{i\omega\lambda}{2}\right)^{1+\frac{i\omega\lambda}{2}} \exp\left[\frac{1}{2}i\omega\lambda^2 y^2\right]\Gamma\left(-\frac{i\omega\lambda}{2}\right){}_1F_1\left(1-\frac{i\omega\lambda}{2};1;-\frac{i\omega\lambda}{2}y^2\right)$$

Mismatch

Mismatch between lensed waveforms with varying M_L and λ and the fiducial waveform with

$M_L = 100 M_\odot,\ \lambda = 1$

To minimize the mismatch we need to align the lensed images by shifting the waveforms by the time delay.

$$\exp(-i\omega t_\lambda) = \exp\left[-i\omega\left(\lambda t - \frac{\lambda(1-\lambda)}{2} y^2\right)\right]$$
$$= \exp\left[-i\omega\lambda\left(t - \frac{y^2}{2}\right)\right] \exp\left(-\frac{1}{2} i\omega\lambda^2 y^2\right)$$

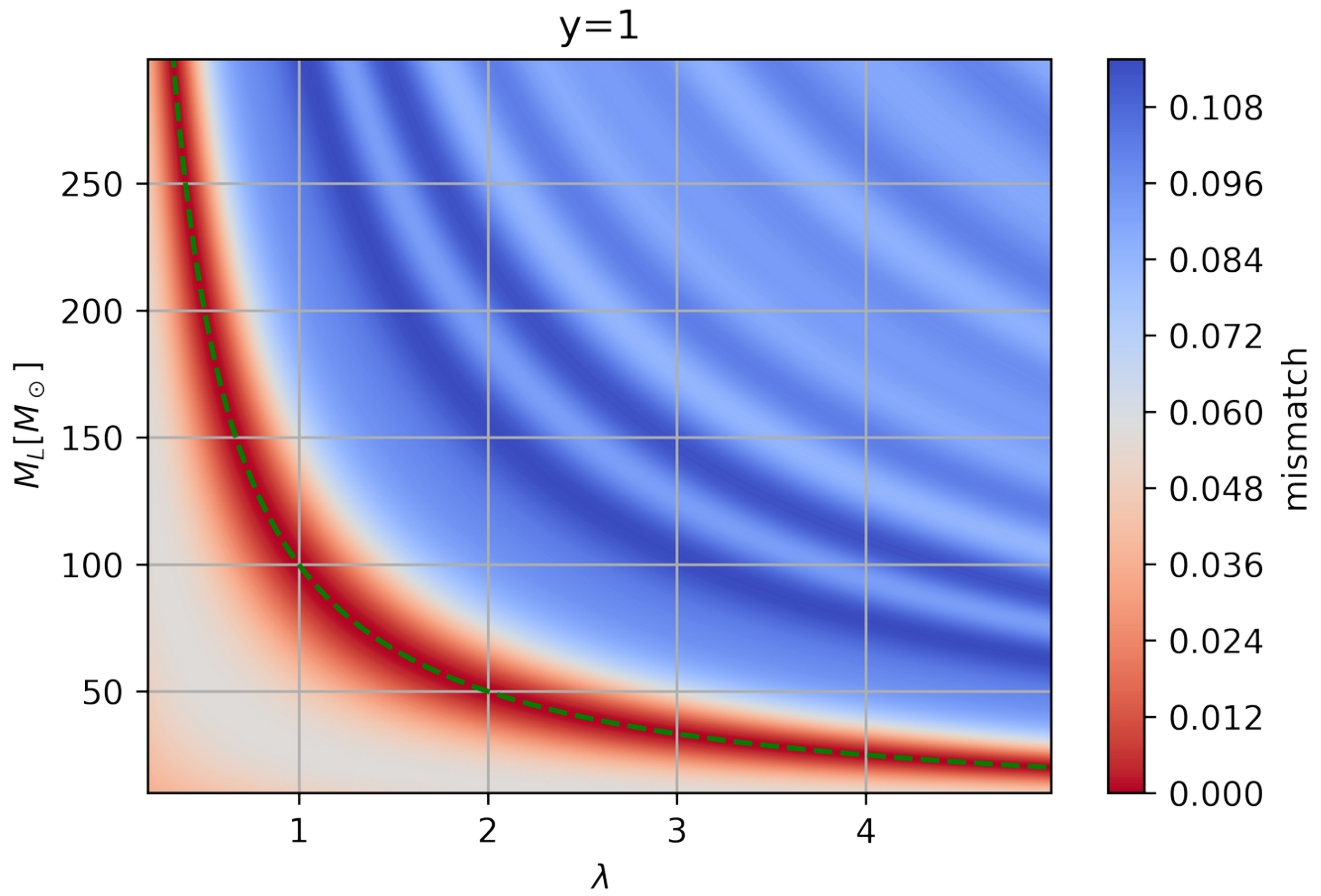
It cancels with

$$F_\lambda = \frac{1}{\lambda}\left(-\frac{i\omega\lambda}{2}\right)^{1+\frac{i\omega\lambda}{2}} \exp\left[\frac{1}{2} i\omega\lambda^2 y^2\right] \Gamma\left(-\frac{i\omega\lambda}{2}\right) {}_1F_1\left(1-\frac{i\omega\lambda}{2}; 1; -\frac{i\omega\lambda}{2} y^2\right)$$

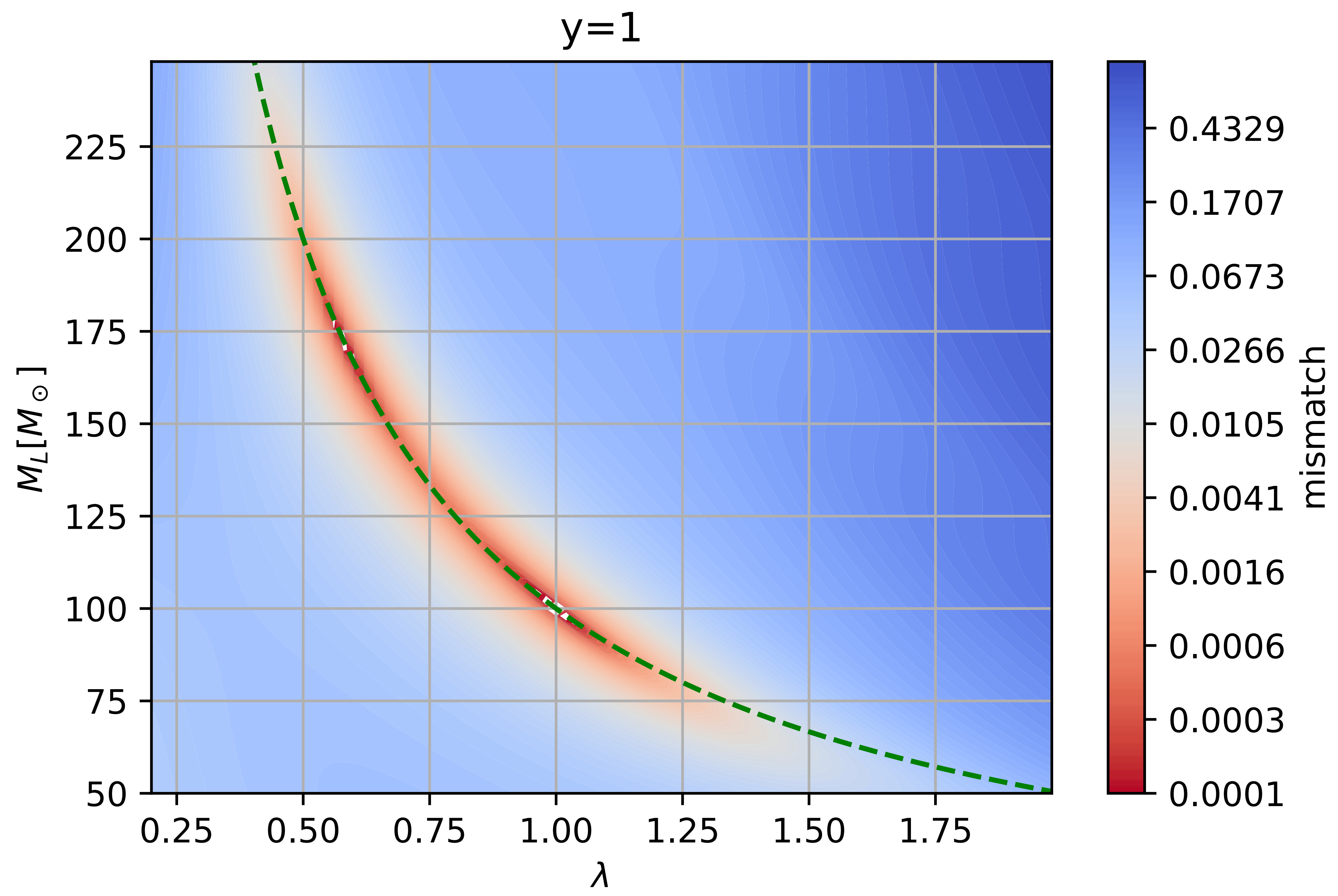
We are left with terms of $\omega\lambda \sim M_L\lambda$ $\qquad \omega = 8\pi(1+z_L) M_L f$

Mismatch

Aligning with time delay with varying M_L and λ

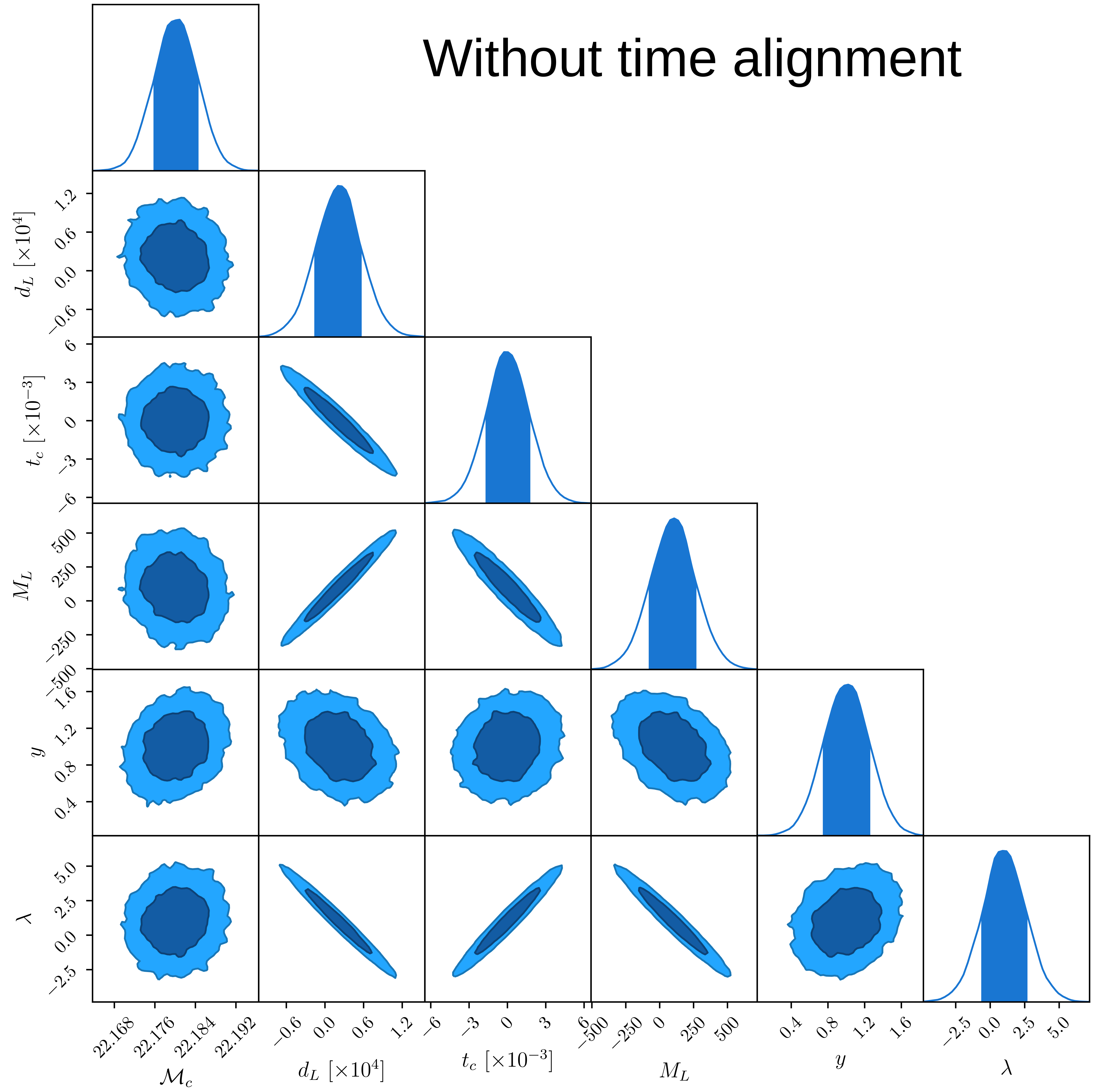


Aligning with time delay with fiducial M_L=100 and λ=1

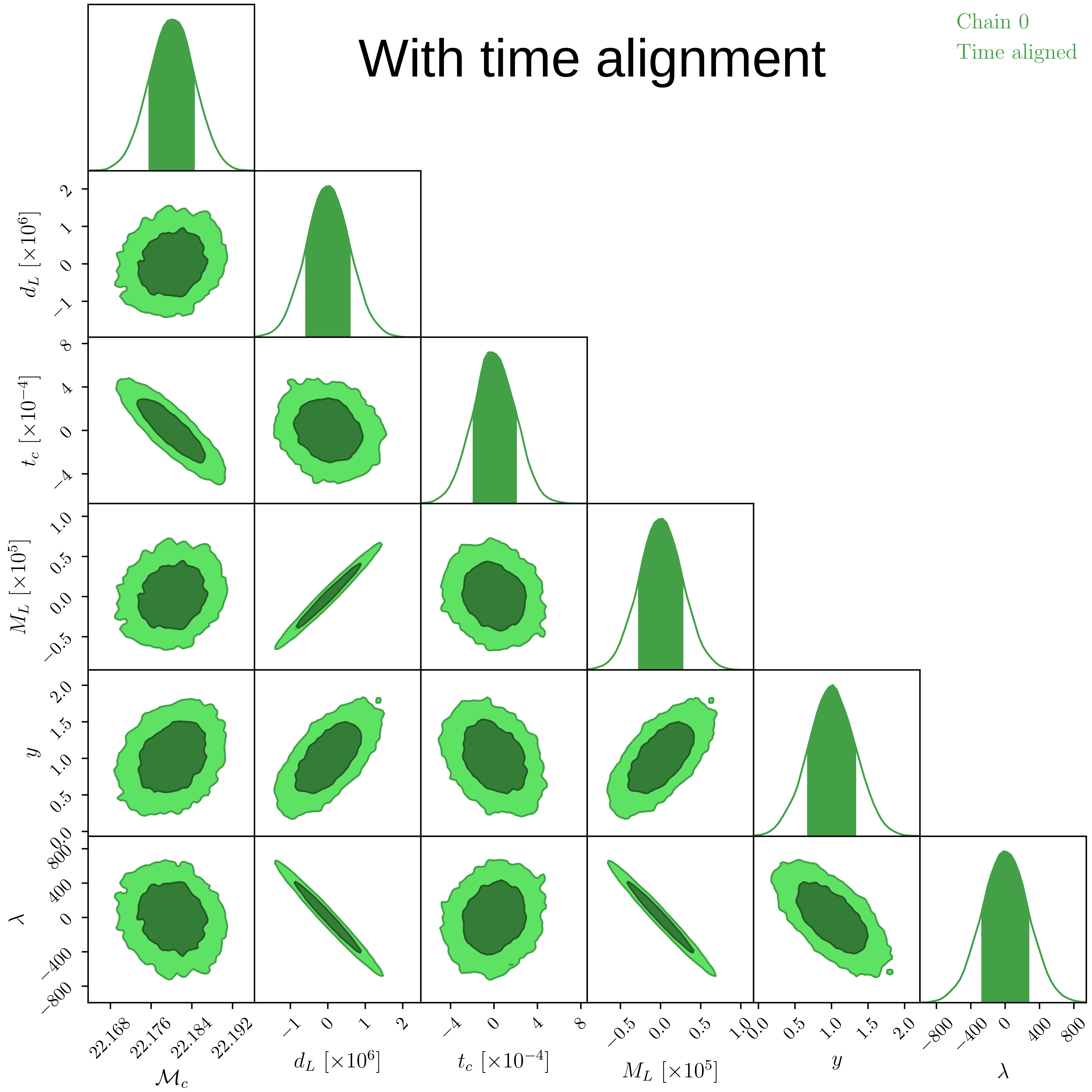


In realistic PE we cannot compute the time delay since we don't know the true values of lensing parameters. We can only estimate the coalescence time t_c.

Fisher matrix forecast



Degeneracy for t_c



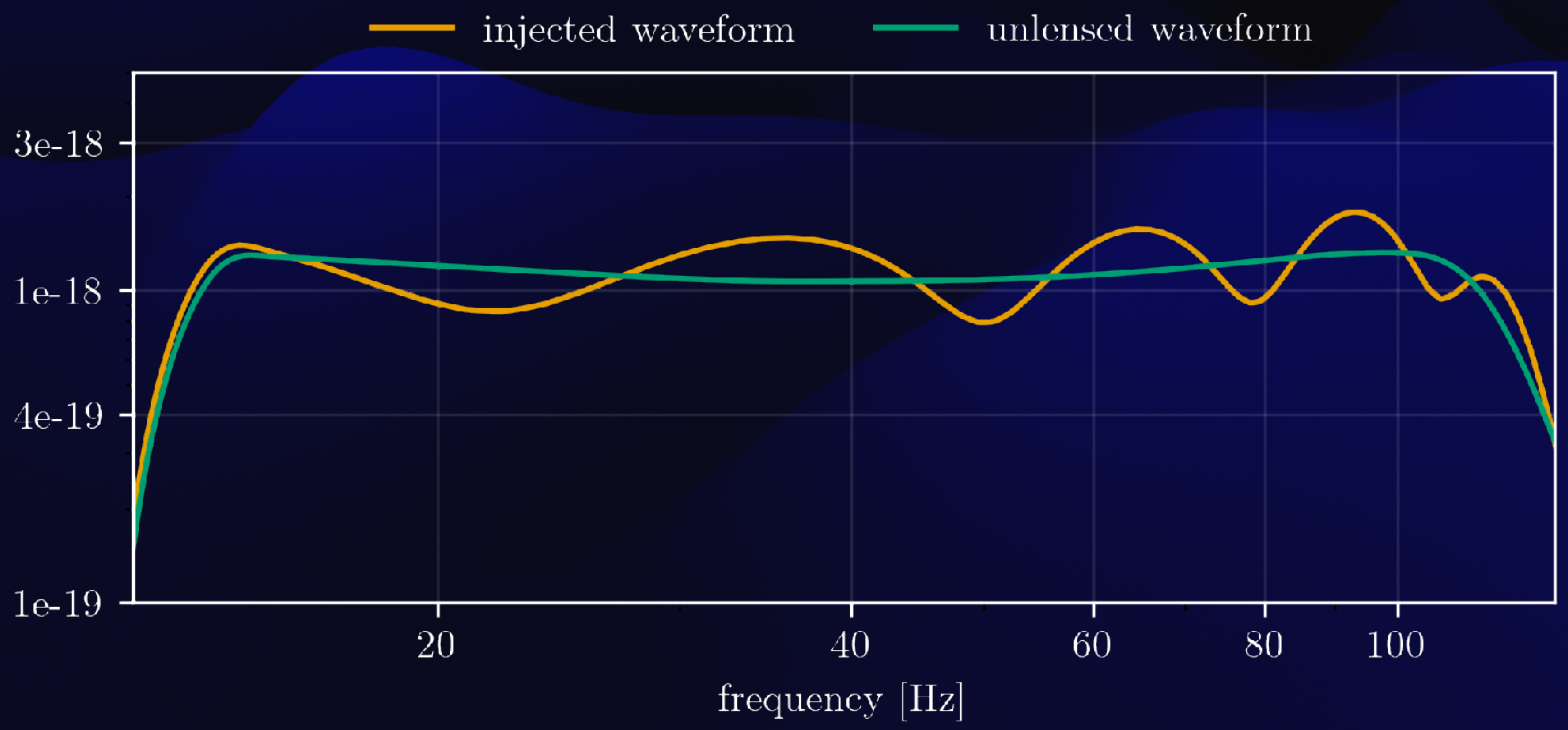
Higher degeneracy for M_L and λ

Mass-Sheet Degeneracy

PE analysis

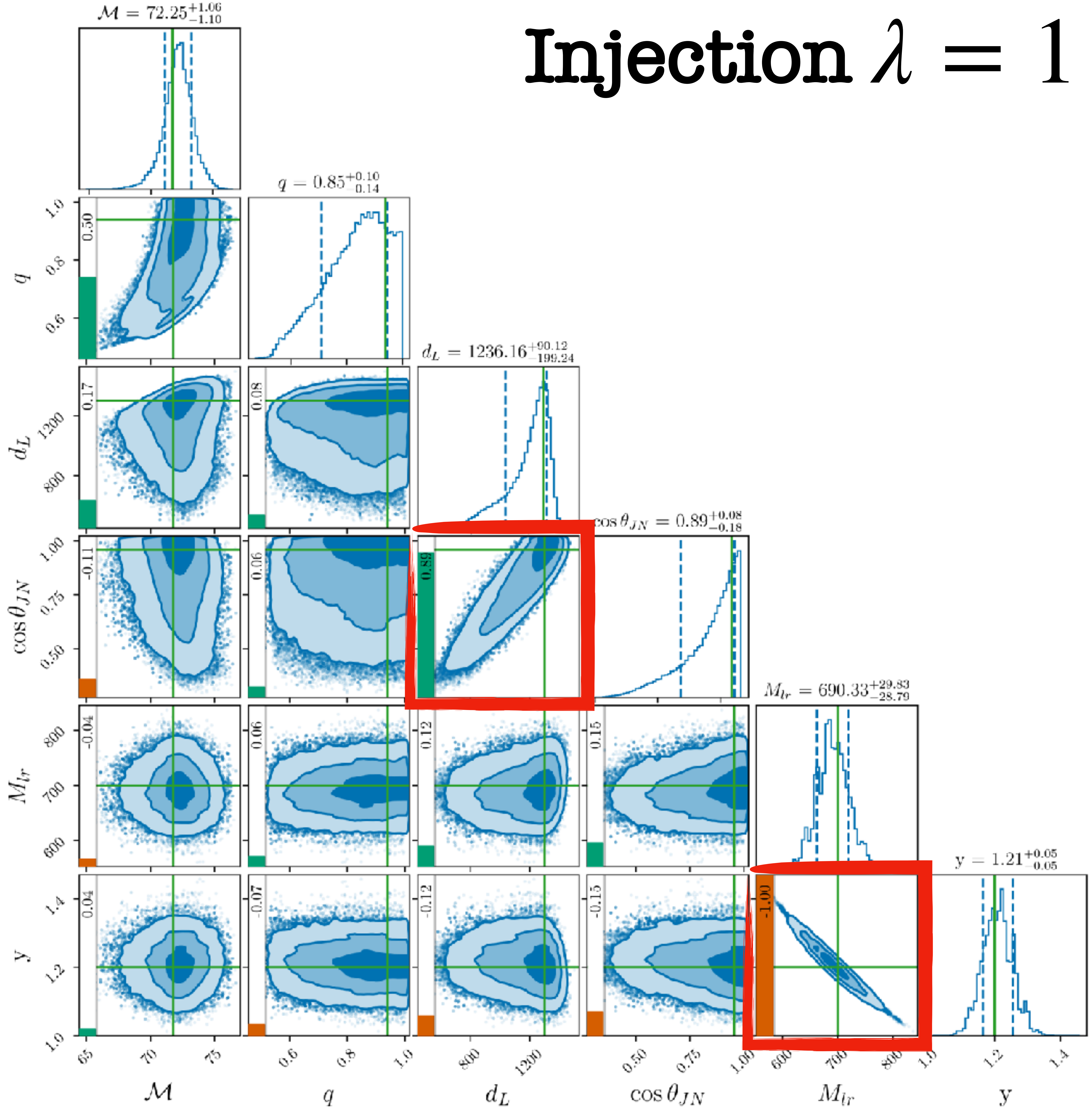
Mass-Sheet Degeneracy

Injection $\lambda = 1$



Parameter	Value
$\mathcal{M}$	71.78
q	0.94
d_L [Mpc]	1300
$\cos\theta_{JN}$	0.95
$M_{l,r}$ $[M_\odot]$	700
y	1.2
λ	1
detectors	H1,L1,V1
optimal SNR	78
wf approx	IMRPhenomXP

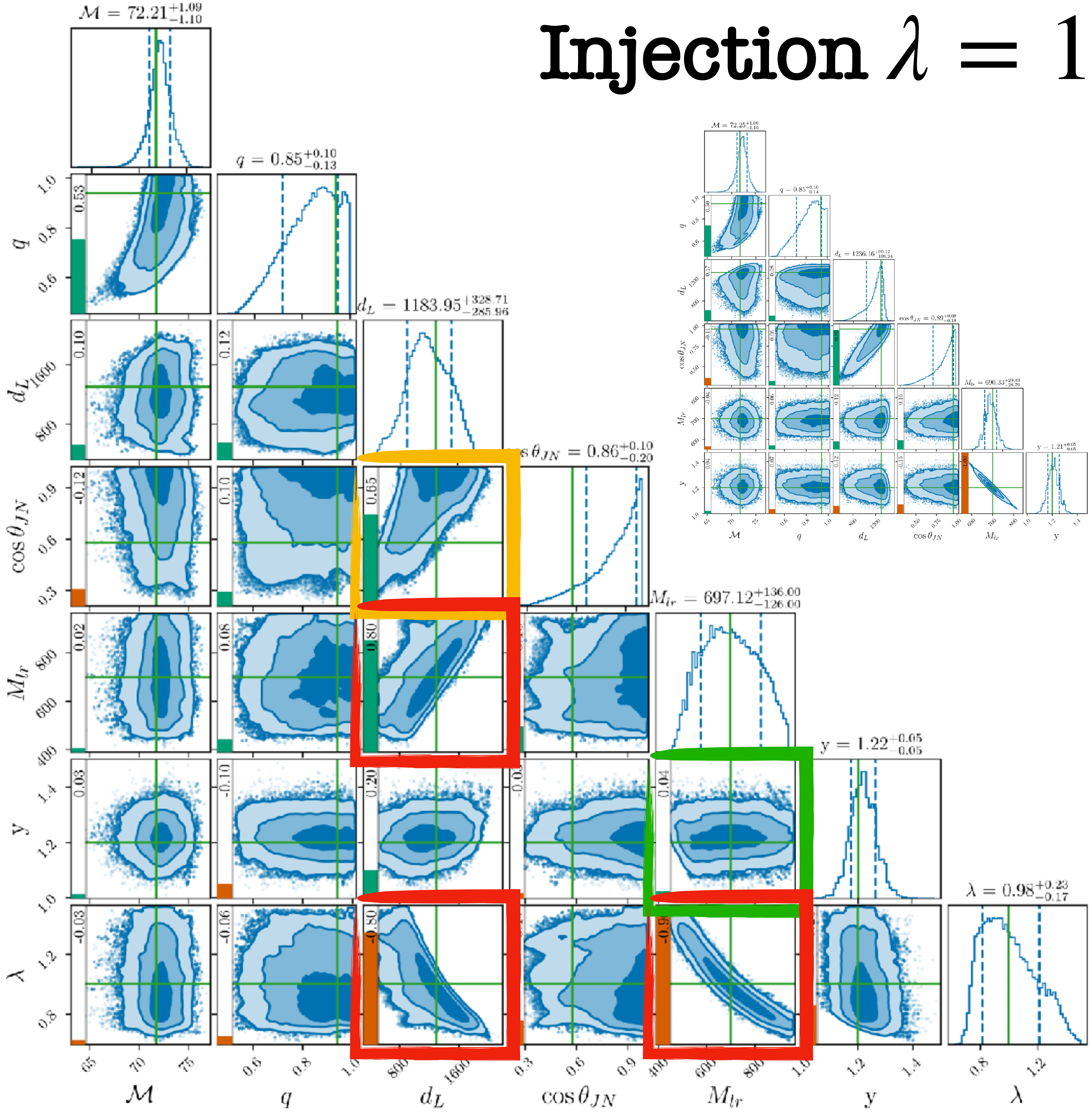
Injection $\lambda = 1$



PE without λ

- High correlation $M_{l,r}$ – y
- High correlation d_L – θ_{JN}

Injection $\lambda = 1$

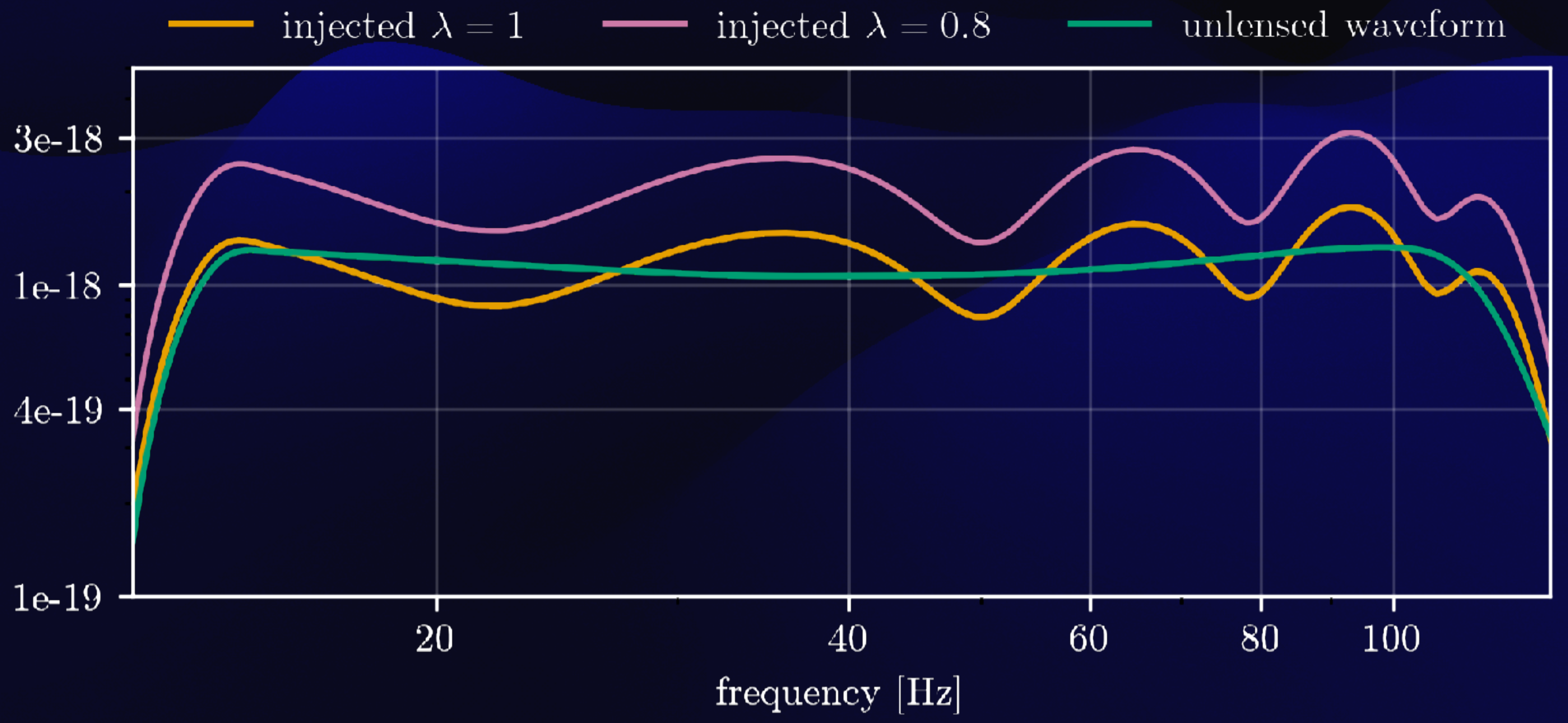


PE with λ

- NO CORR $M_{l,r}$ – y
- correlation to $M_{l,r}$ – λ
- luminosity distance d_L
  - smaller corr w/ θ_{JN}
  - high corr w/ $M_{l,r}$
  - high corr w/ λ

Mass-Sheet Degeneracy

Injection $\lambda = 0.8$



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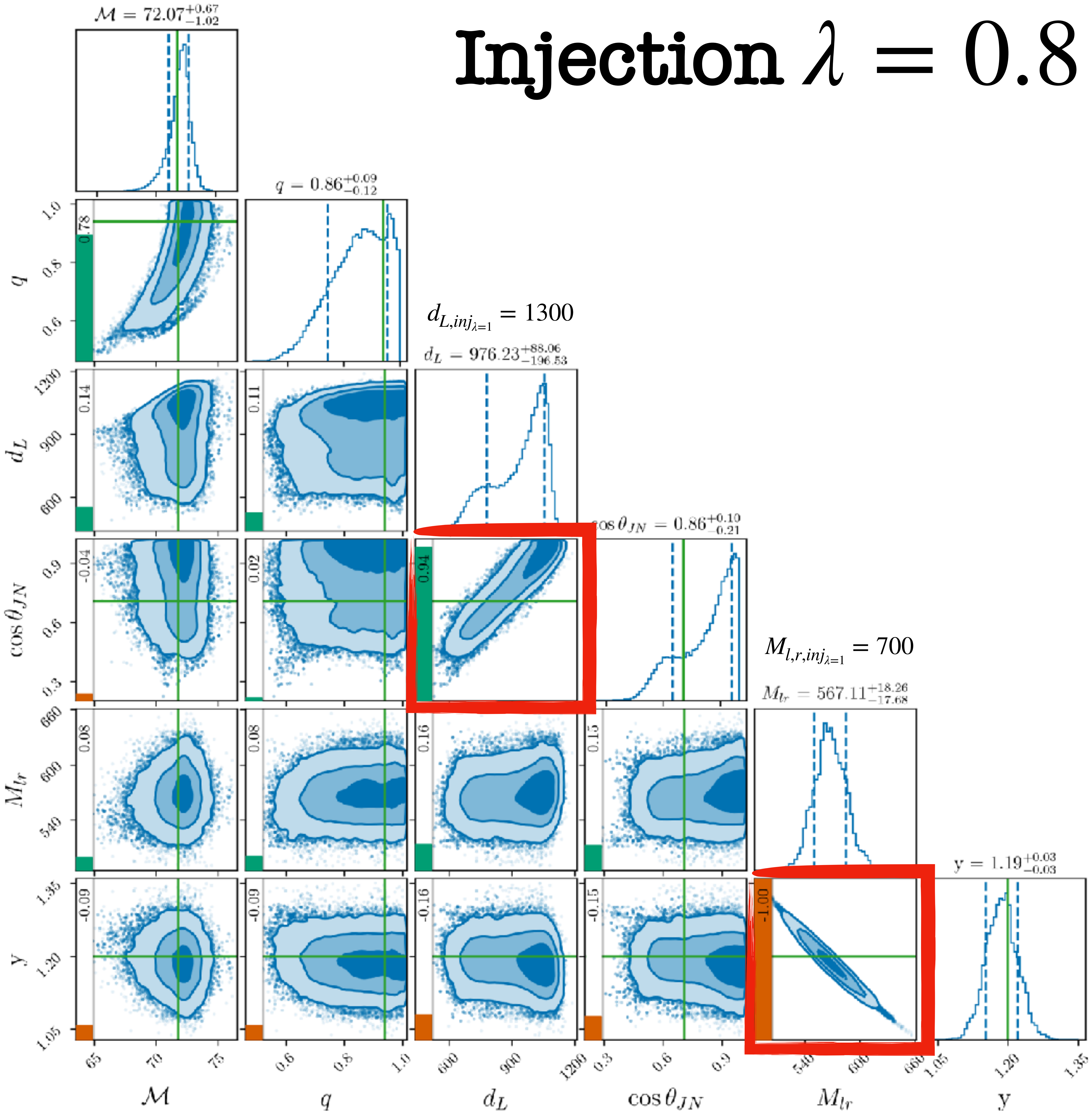
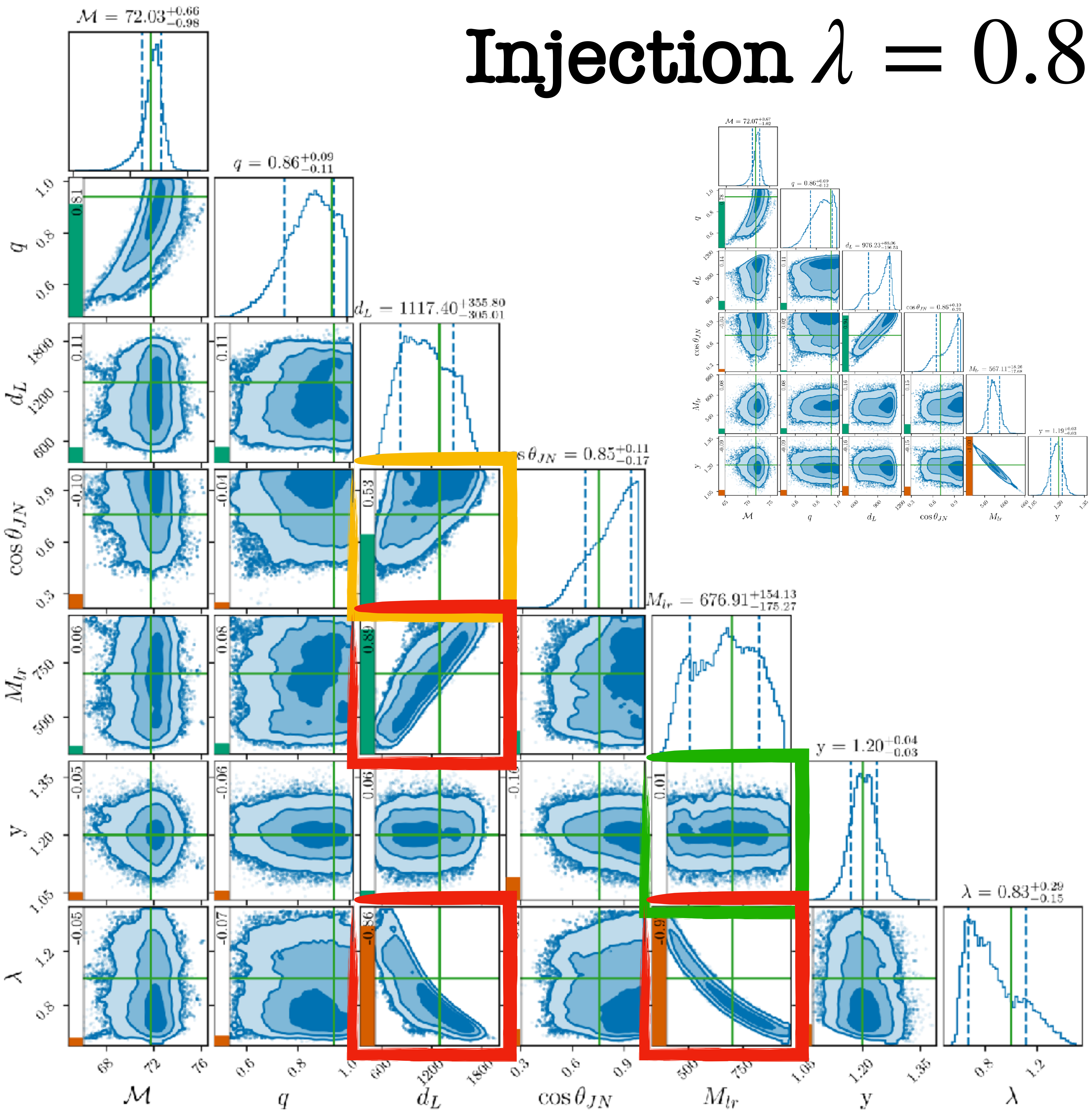


PE without λ

- High correlation $M_{l,r}$ – y
- High correlation d_L – θ_{JN}
- Value of y OK
- Value of $M_{l,r}$ changes
  - absorbs $\lambda = 0.8$, not as expected
  - because of d_L and y

Injection $\lambda = 0.8$

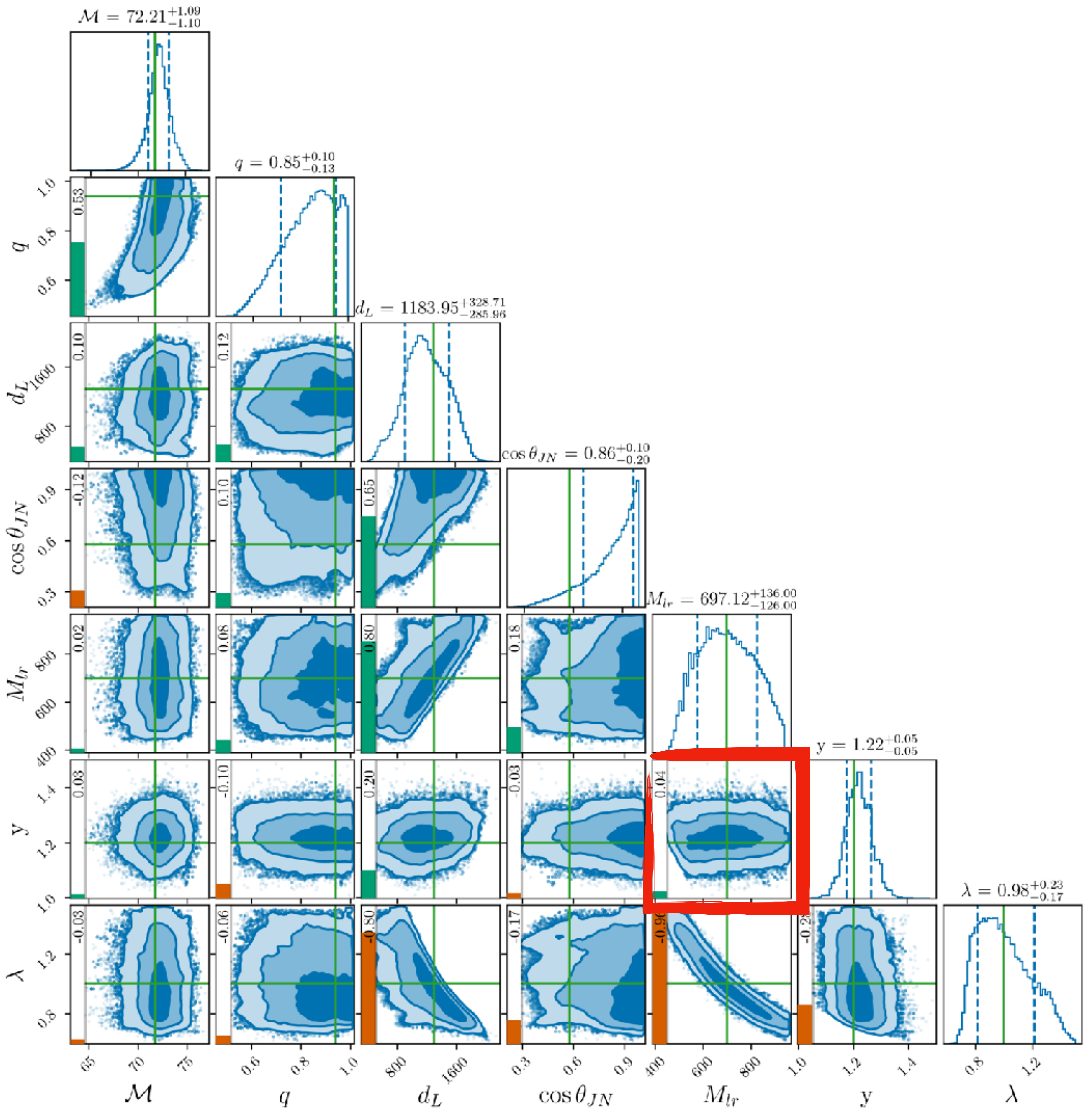


PE with λ

- **All parameters retrieved correctly**
- NO CORR $M_{l,r} - y$
- correlation to $M_{l,r} - \lambda$
- luminosity distance d_L
  - smaller corr w/ θ_{JN}
  - high corr w/ $M_{l,r}$
  - high corr w/ λ

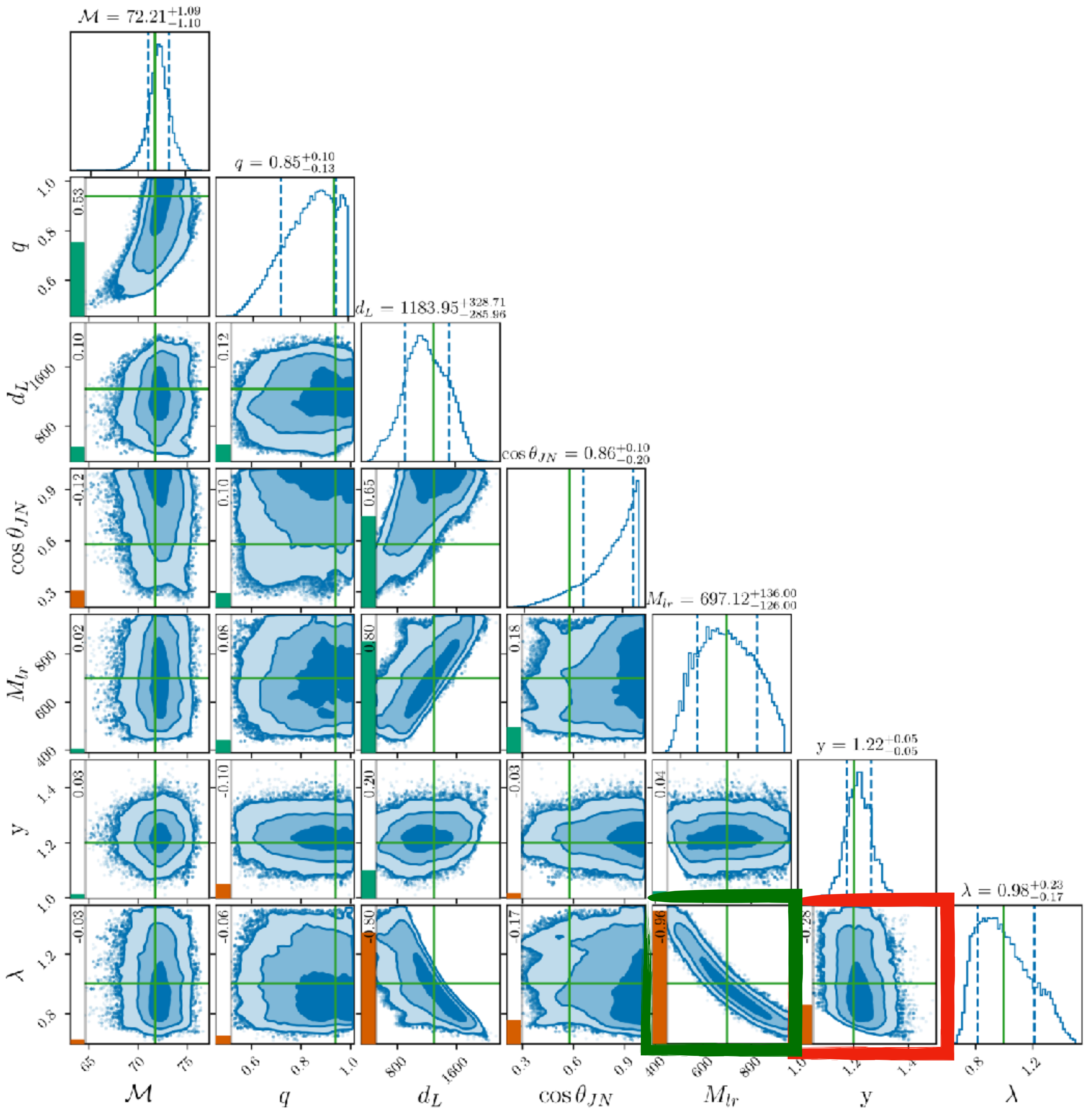
PE summary

- The MSD is parametrised correctly



PE summary

- The MSD is parametrised correctly
- For injections
  - y correlates far less with λ than $M_{l,r}$



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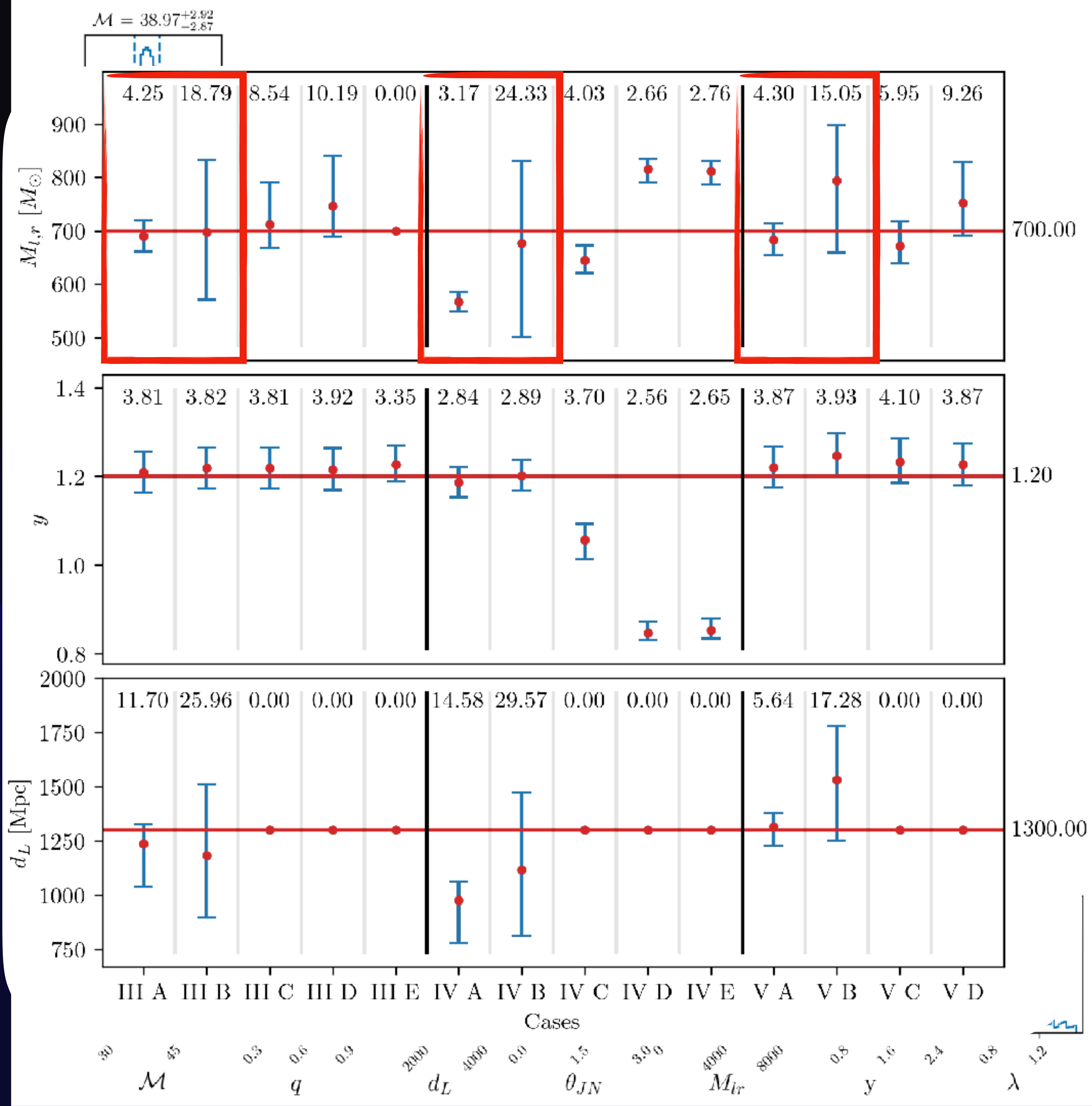
  - considering λ increases $M_{l,r}$ errors



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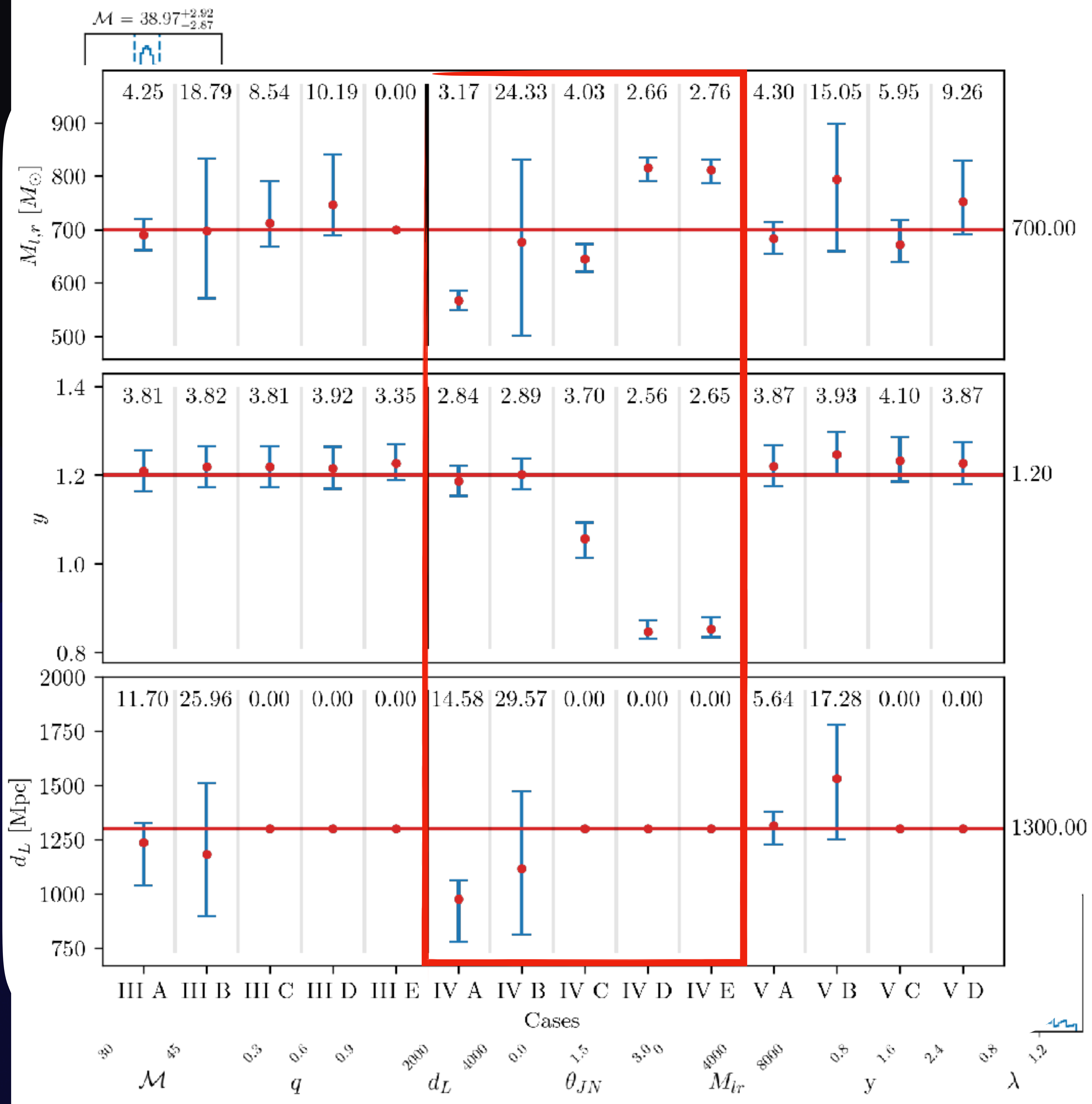
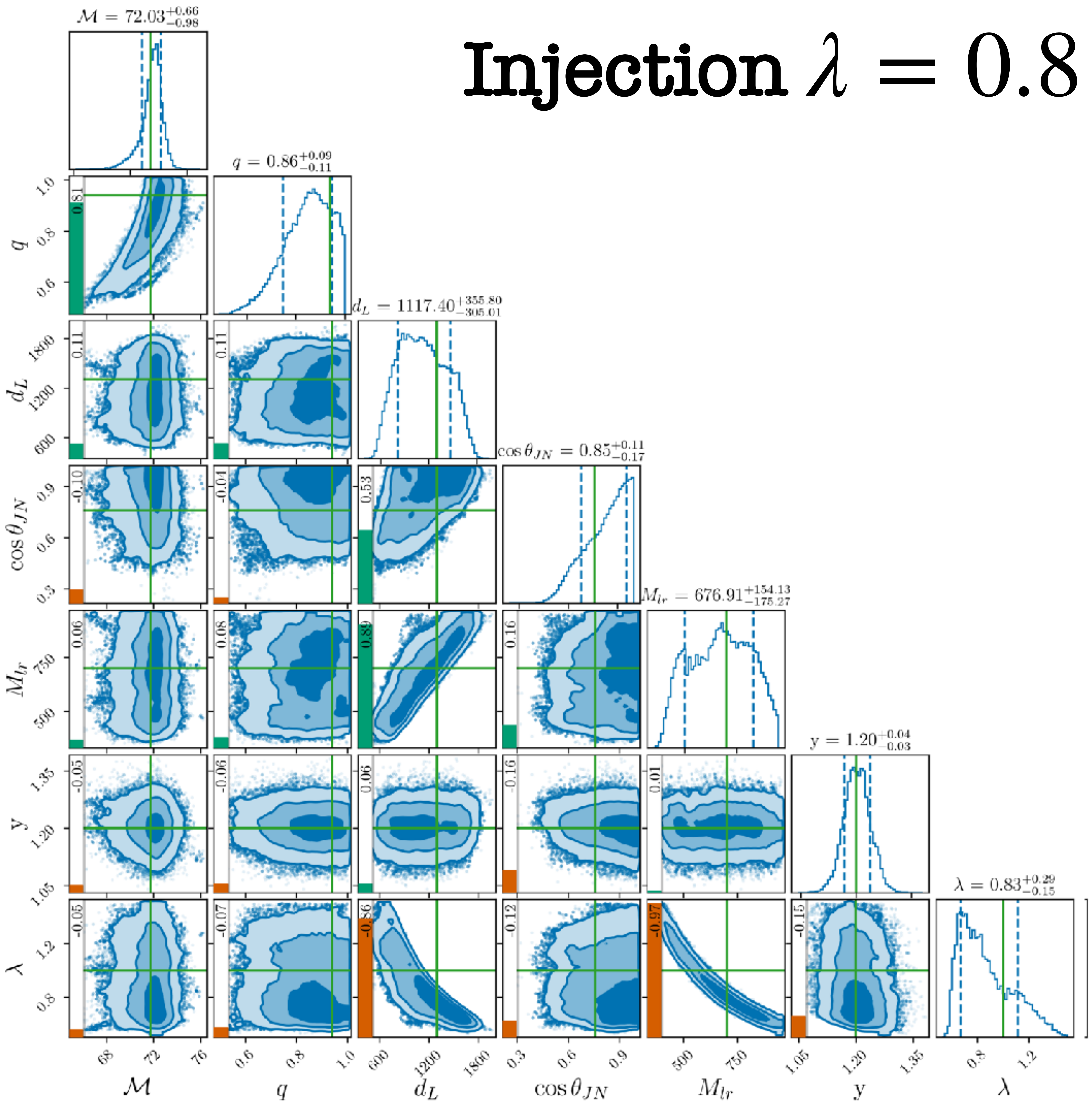
  - the behaviour of $M_{l,r}$ and y depend considerably on other parameters (d_L)



PE summary

- The MSD is parametrised correctly
- For injections
  - y correlates far less with λ than $M_{l,r}$
  - considering λ increases $M_{l,r}$ errors
  - the behaviour of $M_{l,r}$ and y depend considerably on other parameters (d_L)
  - adding λ, we retrive the correct parameters

Injection $\lambda = 0.8$



Open issues

- Role of time shift

$$F_\lambda = \frac{1}{\lambda}\left(-\frac{i\omega\lambda}{2}\right)^{1+\frac{i\omega\lambda}{2}} \boxed{\exp\left[\frac{1}{2}i\omega\lambda^2 y^2\right]} \Gamma\left(-\frac{i\omega\lambda}{2}\right) {}_1F_1\left(1-\frac{i\omega\lambda}{2};1;-\frac{i\omega\lambda}{2}y^2\right)$$

- What happens in the geometric limit
- Increase width of prior in PE analysis
- How to translate the problem to constraining H_0